Multiple Machines

- **Model Multiple Available resources**
  - **people**
  - **time slots**
  - **queues**
  - **networks of computers**
- **Now concerned with both allocation to a machine and ordering on that machine.**

$P||C_{\max}$

NP-complete from partition.

Example

j	p_j
1	**10**
2	**8**
3	**6**
4	**4**
5	**2**
6	**1**

- **What is the makespan on 2 machines?**
- **3 machines ?**
- **4 machines ?**

Approxmiation Algorithms

- **Cannot come up with an optimal solution in polynomial time**
- **Will look at relative error :** $C_{\max}$(**our algorithm**)$/C_{\max}(OPT)$
- **Challenges:**
  - **Our algorithm's performance is different on different instances**
  - **We can't compute** $C_{\max}(OPT)$

Approxmiation Algorithms

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- **Challenges:**
  - **Our algorithm's performance is different on different instances**
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Solution:

- **We will use a worst case measure on performance**
- **We will use a lower bound on** $C_{\max}(OPT)$

Approximation Algorithms

An algorithm A is a ρ approximation algorithm for a problem, if for all inputs

$$\frac{C_{\max}(A)}{C_{\max}(OPT)} \leq \rho$$

.

In addition, A must run in polynomial time.

We can't compute $C_{\max}(OPT)$.

Recipe:

- **Instead, we compute a lower bound $LB(OPT)$, such that**
  - $LB(OPT)$ **is easy to compute**
  - $LB(OPT) \leq C_{\max}(OPT)$.
- **We then show that $C_{\max}(A) \leq \rho LB(OPT)$.**

Combining the previous two steps, we have:

$$C_{\max}(A) \leq \rho LB(OPT) \leq \rho C_{\max}(OPT)$$

which can be rewritten as

$$\frac{C_{\max}(A)}{C_{\max}(OPT)} \leq \rho$$

.

Notes:

- **Must come up with a good lower bound**
- **Can replace** $C_{\max}$ **with any objective.**

Lower Bounds for $P||C_{\max}$

- **Average load**
- **Longest job**

Lower Bounds for $P||C_{\max}$

- **Average load** – $\lceil \Sigma\, p_j/m \rceil$
- **Longest job** – $p_{\max} = \max_j\{p_j\}$

List Scheduling Algorithm

A Greedy Algorithm

1. Make a list of the jobs (in any order)
2. When a machine becomes available, schedule the next job on the list.

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1. **Make a list of the jobs (in any order)**
2. **When a machine becomes available, schedule the next job on the list.**

Analysis

- **Let** t **be the last time at which all machines are busy.**
- $t \leq \Sigma_j \, p_j/m$
- $C_{\max} \leq t + p_{\max} \leq \Sigma_j \, p_j/m + p_{\max}$.

Put this together with our lower bound:

$$C_{\max} \leq t + p_{\max} \leq \sum_j p_j/m + p_{\max} \leq 2LB \leq 2OPT$$

Improved Algorithm

- **Schedule length is average load plus last job.**
- **When last job is small, the schedule is shorter.**
- **Force last job to be small – LPT (Longest Processing Time).**

LPT is a 4/3-approximation for $P||C_{\max}$.

Proof Outline

- **If last job is small ($\leq 1/3OPT$) then 4/3-approximation**
- **Otherwise, there are at most 2 jobs per machine and LPT is optimal.**

Even better algorithms are possible: . A polynomial-time approximation scheme (PTAS) is an algorithm that, given fixed $\epsilon > 0$, returns at $(1+\epsilon)$ -approximation in polynomial time. The running time can have a bad dependence on ϵ, such as $n^{O(1/\epsilon)}$.

$P||C_{\max}$ **has a PTAS.**

Precedence Constraints

- $P\infty|\text{prec}|C_{\max}$ **is known as project scheduling.**
- $P|\text{prec}|C_{\max}$ **has a 2-approximation.**

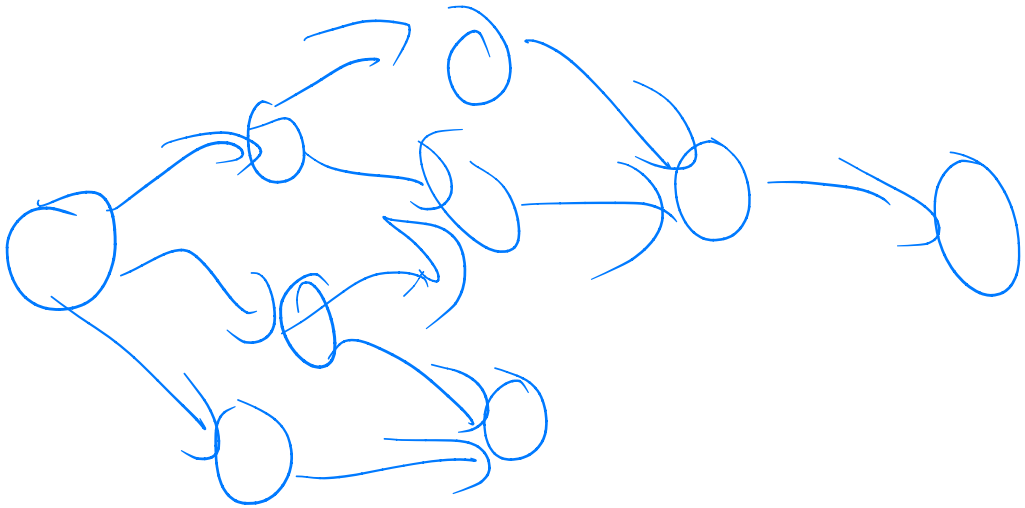
What are good lower bounds for $P|\text{prec}|C_{\max}$ **?**

Precedence Constraints

- $P\infty|\text{prec}|C_{\max}$ **is known as project scheduling.**
- $P|\text{prec}|C_{\max}$ **has a 2-approximation.**

What are good lower bounds for $P|\text{prec}|C_{\max}$ **?**

- **Average load**
- $p_{\max}$
- **any path in the precedence graph**
- **the critical path is the longest path in the precedence graph.**

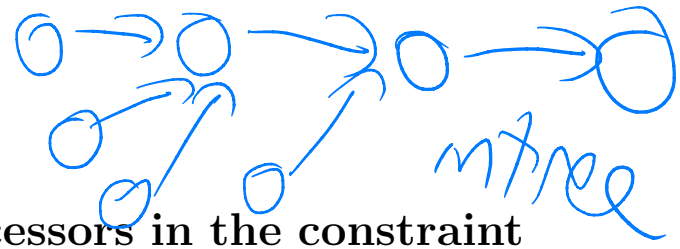


Unit Processing Times

$P|p_j = 1, \text{prec}|C_{\max}$ **is NP-hard.**

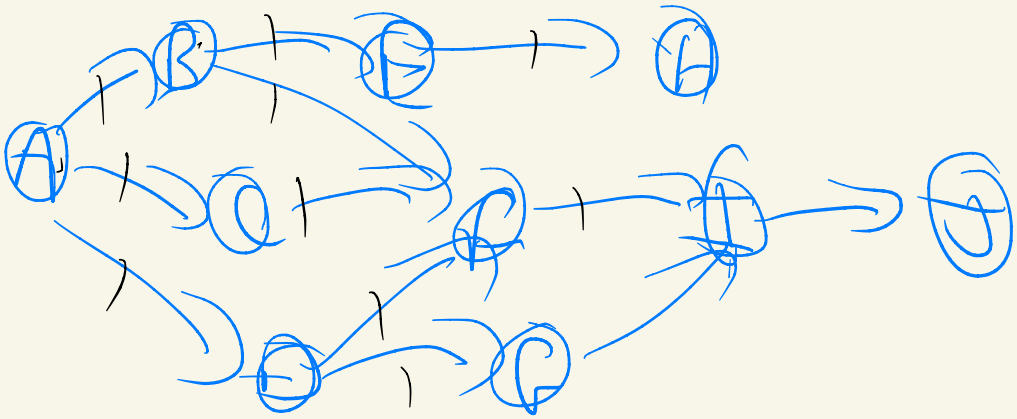
Heuristics

- **Critical Path (CP) rule**
  - **The job at the head of the longest string of jobs in the constraint graph has the highest priority**
  - $P|p_j = 1, tree|C_{\max}$ **is solved by CP.**
- **Largest Number of Successors First (LNS)**
  - **The job with the largest total number of successors in the constraint graph has highest priority.**
  - **For in-trees and chains, LNS is identical to CP**
  - **LNS is also optimal for** $P|p_j = 1, outtree|C_{\max}$
- **Generalization to arbitrary processing times is possible**



Fixed Number of Processors

- $P2|p_j = 1, \text{prec}|C_{\max}$ **is solvable in polynomial time**
- $P3|p_j = 1, \text{prec}|C_{\max}$ **is a big open question.**



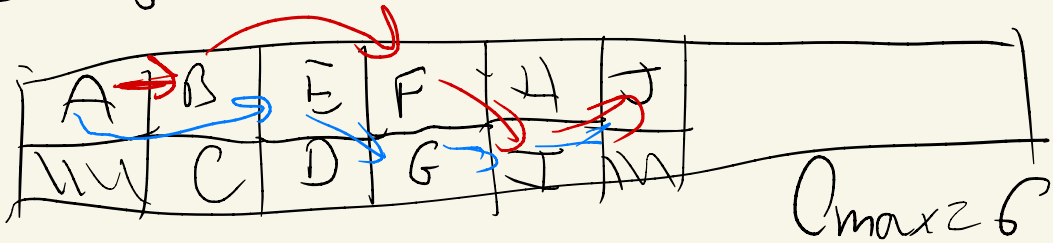
2 machines

A	B	E	H	G	I	J	
///	C	D	F	///	///	///	

$LB = \frac{\sum P_i}{m} = \frac{10}{2} = 5$

$C_{max} \geq$

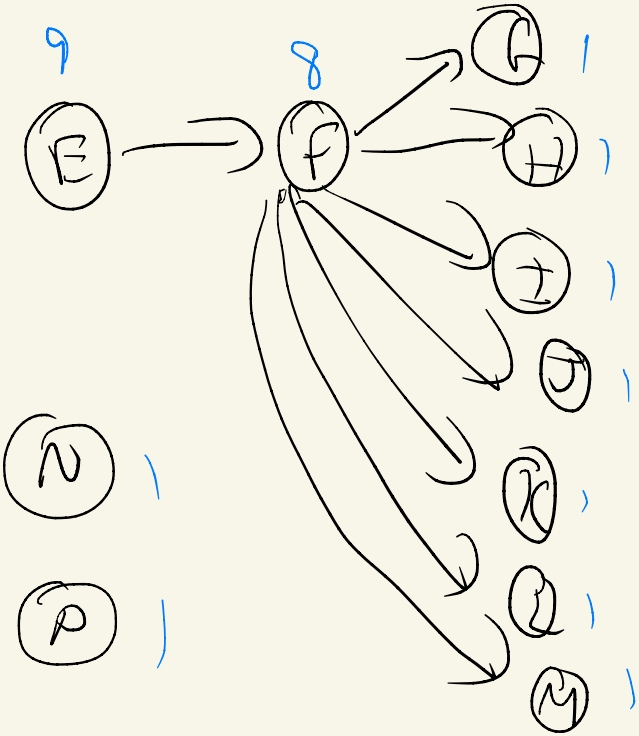
$CP = 6$



Good idea

- work on jobs on critical paths
- jobs w/ large #s of successors

#successor

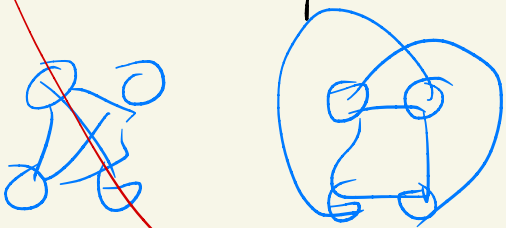


$P3 \mid prec, p_j = 1 \mid C_{max}$

Garey + Johnson NP-complete
~1978

13 open problems

- Linear Programming (P)
- Graph Isomorphism



2020 1 unresolved problem

Preemptions: $P|\text{pmtn}|C_{\max}$

- **McNaughton's wrap-around rule is optimal.**

LB $\frac{\sum p_j}{m}$ $\frac{31}{3}$

$m = 3$

Example

j	p_j
A	**7**
B	**10**
C	**1**
D	**4**
E	**9**

$p_{max} = 10$

$\max\left(p_{max}, \frac{\sum p_j}{m}\right)$ is a lower bound

Simple alg. for this problem

Preemptions: $P|\text{pmtn}|C_{\max}$

- **McNaughton's wrap-around rule is optimal.**

$m=3$

$LB=\max\left(\frac{31}{3}, 10\right)=\frac{31}{3}$

Example

j	p_j
A	**7**
B	**10**
C	**1**
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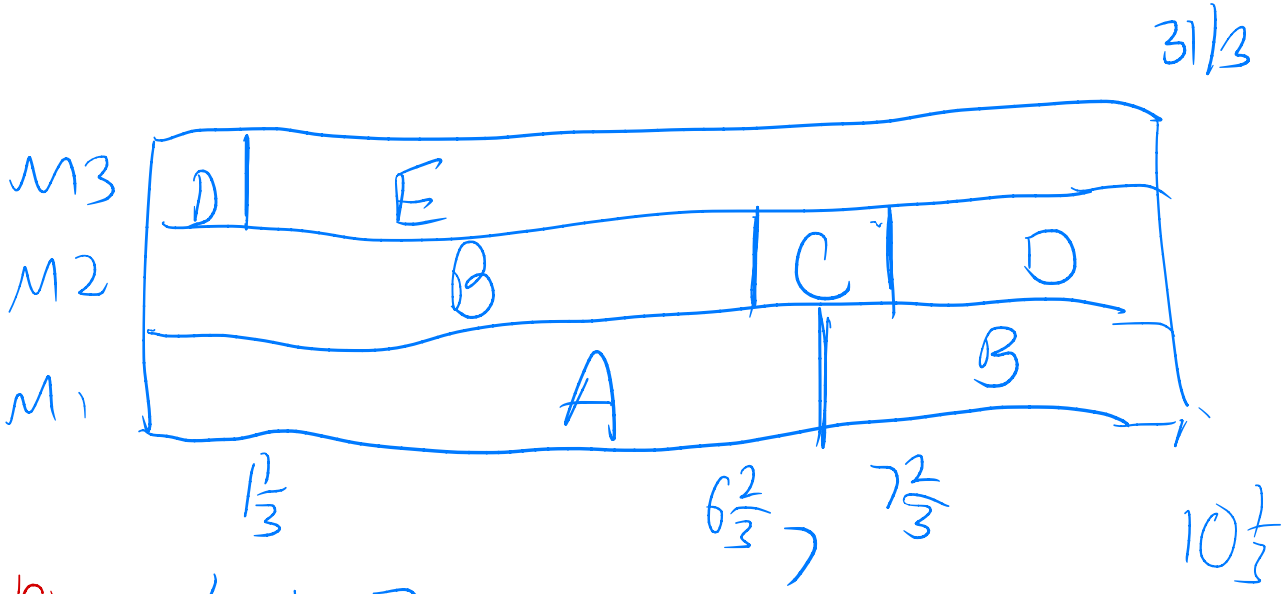
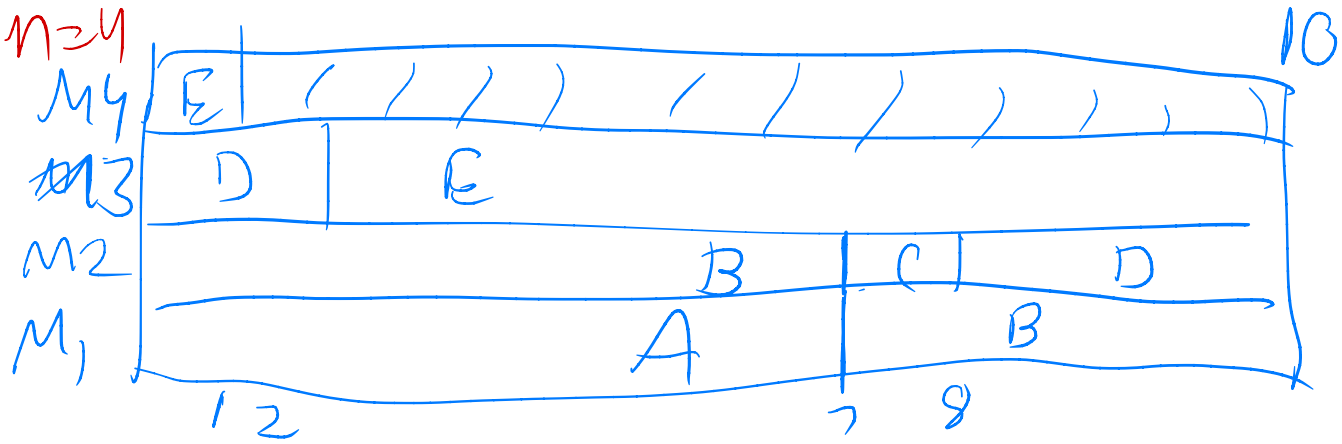


$LB=\max\left(\frac{31}{4}, 10\right)=10$ 達成



Preemptions: $P|\text{pmtn}|C_{\max}$

- **McNaughton's wrap-around rule is optimal.**

Example

j	p_j
A	**7**
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C	**1**
D	**4**
E	**9**

Correctness

- Jobs have to fit
- Valid schedule: no job is on ≥ 2 machines at the same time.

time allowed $\geq \frac{\sum p_j}{m} \cdot m = \sum p_j$

time $\geq p_{\max}$

LP for $P|\text{pmtn}|C_{\max}$

amount of

Variables: x_{ij} **is the time that job** j **runs on machine** i **.** $C_{\max}$ **is also a variable.**

Constraints

- **Each job runs for** p_j **units of time**
- **Each machine runs for at most** $C_{\max}$ **time.**
- $C_{\max}$ **is more than any processing time.**

$$\min C_{\max} \tag{1}$$

$$s.t. \tag{2}$$

$$\Sigma_{i=1}^{m} x_{ij} = p_j \quad j = 1 \ldots n \tag{3}$$

$$\Sigma_{j=1}^{n} x_{ij} \leq C_{\max} \quad i = 1 \ldots m \tag{4}$$

$$\Sigma_{i=1}^{m} x_{ij} \leq C_{\max} \quad j = 1 \ldots n \tag{5}$$

$$x_{ij} \geq 0 \quad \forall i,j \tag{6}$$

Note that LP only assigns pieces of jobs to machines. Need to also assign jobs to times.

corresp. to example →

$x_{1A} = 7$ $x_{2B} = 6\frac{2}{3}$ $x_{2D} = 2\frac{2}{3}$ $x_{3E} = 9$

$x_{1B} = 3\frac{1}{3}$ $x_{2C} = 1$ $x_{3D} = 1\frac{1}{3}$

Machines with speeds – $Q|\text{pmtn}|C_{\max}$

- **Machines** $M_1, \ldots, M_m$ **with speeds** $v_1, \ldots, v_m$ **.**
- **Assume wlog that** $v_1 \geq v_2 \geq v_m$
- **Assume wlog that** $p_1 \geq p_2 \geq p_n$
- **If a job runs for one unit of time on machine** M_i **, it uses up** v_i **units of processing.**
- **If job** j **runs on machine** M_i **, then it takes** p_j/v_i **time units to complete.**

Example

j	p_j
A	**20**
B	**16**
C	**2**
D	**1**

3 machines

$v_1 = 4$

$v_2 = 2$

$v_3 = 1$

What are the lower bounds

Job A can run

2 units of time on M_1,

$20 - 4(2) = 12$

+ 6 units of time on M_2

Lower bounds for $Q|\text{pmtn}|C_{\max}$

- **What is the analog of $p_{\max}$?**
- **What is the analog of average load ?**
- **Are there others ?**

P_i / v_i (biggest job on fastest machine)

$\frac{\sum_j P_j}{\sum_i v_i}$ (total processing over sum of speeds)

$\frac{P_1}{v_1} = \frac{20}{4} = 5$

$\frac{\sum P_j}{\sum v_i} = \frac{39}{7} \rightarrow$

7

Example

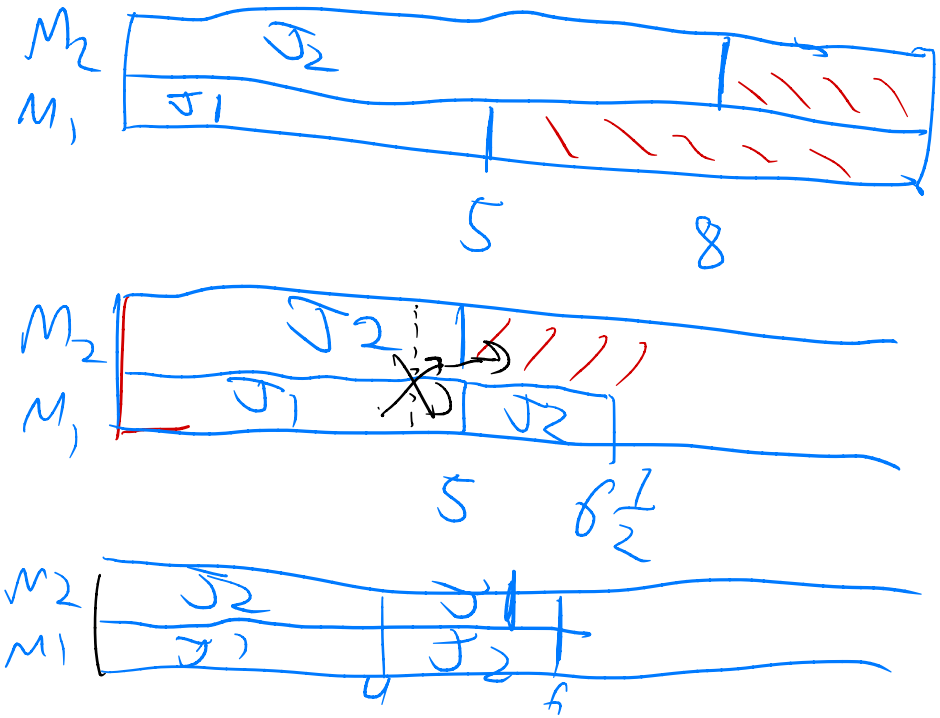
$P_1 = 20$

$P_2 = 16$

$v_1 = 4$

$v_2 = 2$

OPT

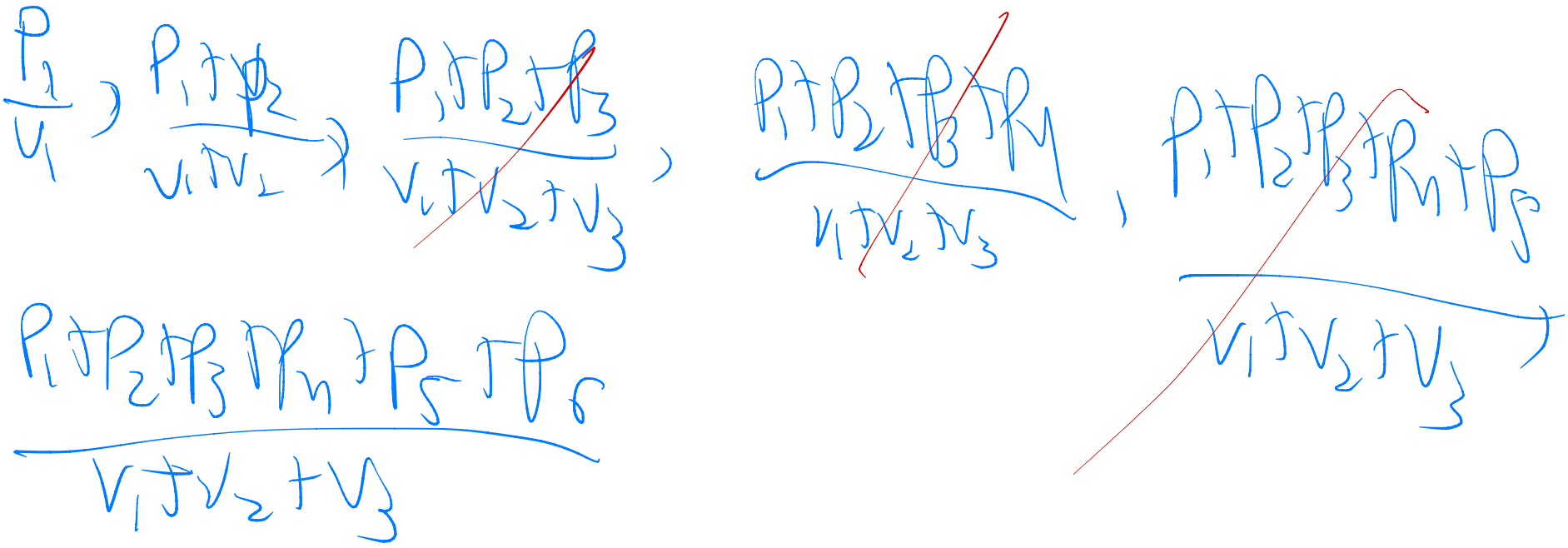


Lower bounds for $Q|\text{pmtn}|C_{\max}$

- **What is the analog of** $p_{\max}$ **?** – p_1/v_1
- **What is the analog of average load ?** – $\Sigma\, p_j / \Sigma\, v_i$
- **Are there others ? – Yes**

General Lower Bound

$$C_{\max} \geq \max\left(\frac{p_1}{v_1}, \frac{p_1+p_2}{v_1+v_2}, \ldots, \frac{\Sigma_{j=1}^{m-1} p_j}{\Sigma_{i=1}^{m-1} v_i}, \frac{\Sigma_{j=1}^{n} p_j}{\Sigma_{i=1}^{m} v_i}\right)$$



Lower Bound

$$C_{\max} \geq \max\left(\frac{p_1}{v_1}, \frac{p_1+p_2}{v_1+v_2}, \ldots, \frac{\Sigma_{j=1}^{m-1} p_j}{\Sigma_{i=1}^{m-1} v_i}, \frac{\Sigma_{j=1}^{n} p_j}{\Sigma_{i=1}^{m} v_i}\right)$$

What is the lower bound for our example?

Can we achieve this lower bound?

$p = (20, 16, 9, 1)$

$v = (4, 2, 1)$

$C_{\max} \geq \max\left(\frac{20}{4}, \frac{20+16}{4+2}, \frac{20+16+9+1}{4+2+1}\right)$

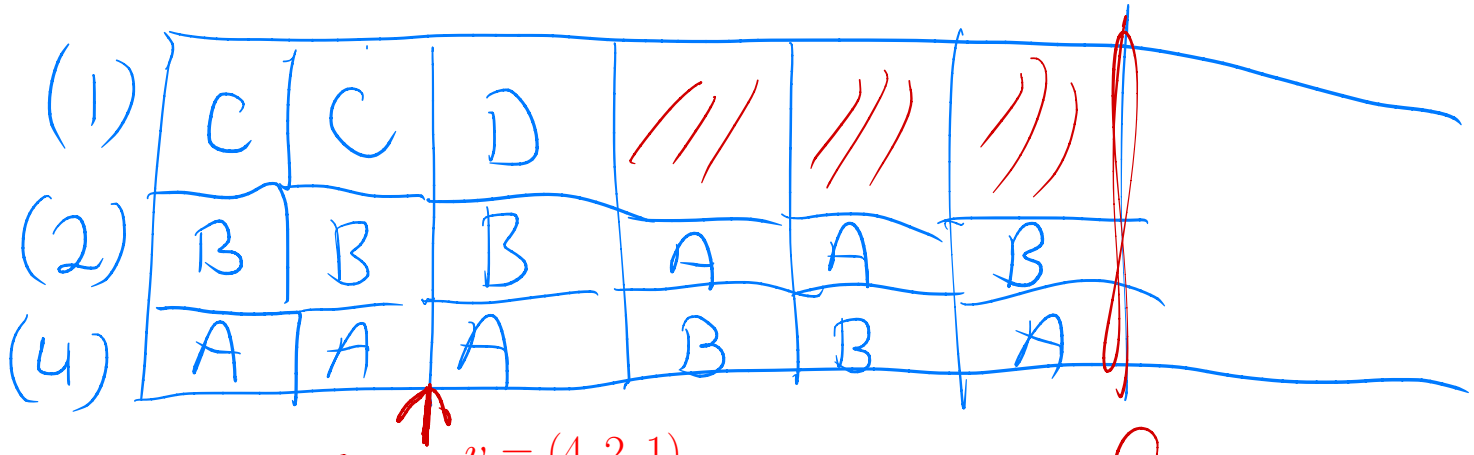
$\max\left(5, 6, \frac{46}{7}\right)$

LRPT-FM

Longest Remaining Processing Time on Fastest Machines

Example 1

j	p_j
A	20
B	16
C	2
D	1



Example 2

j	p_j
A	20
B	16
C	12
D	1

Notes:

- **LRPT-FM is optimal in continuous time**
- **LRPT-FM is near otimal in discrete time, for small time steps.**

LRPT-FM

Longest Remaining Processing Time on Fastest Machines

$C_{max} \geq \left(\frac{20}{4}, \frac{20+16}{4+2}, \frac{20+16+12+1}{4+2+1} \right)$

$= 5, 6, \frac{49}{7} = 7$

Example 1

j	p_j
A	20
B	16
C	2
D	1

A 20 16 12 8 7 3 2 1
B 16 14 12 10 6 5 3
C 12 11 10 9 7 5
D 1 1 1 1 1

$v = (4, 2, 1)$

C	C	C	A	B	A	B
B	B	B	C	C	C	A
A	A	A	B	A	B	C D

(2) (4)

Example 2

j	p_j
A	20
B	16
C	12
D	1

4 2 1

Notes:

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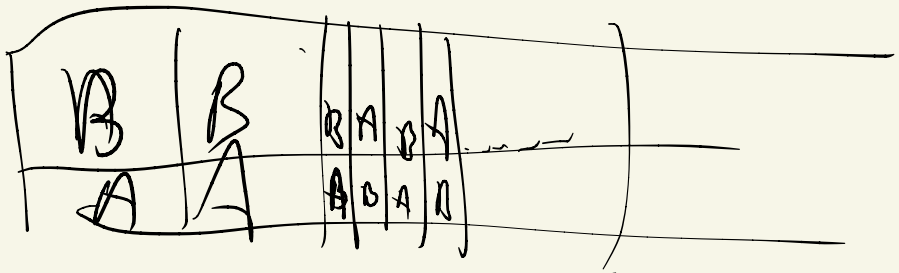
Continous time



2 ⟶ 3

switch infinitely often between time 2 & 3